Homer Simpson and his Doughnuts

Paul McDonald, Kirkcaldy High School

I have always been inspired by the writings of Marcus Aurelius, the famous Roman emperor who said of each particular thing, ask what is it in itself? What is its nature? It has prompted me to explore my own understanding of our magnificent subject. In class, we are all keen to challenge our pupils in order to encourage them to think and talk about their ideas; not always easy with 21st century kids who expect us to entertain them on demand. Interestingly, I have always found TV a marvellous tool for teaching enrichment activities. It can turn a dull topic for some into an exciting and challenging experience.

So who better than Homer Jay Simpson, a fictional character in the animated television series, *The Simpsons*, to help introduce some facts regarding the doughnut or 'torus', one of the most fundamental mathematical objects in the universe. Homer has legendary status and a cult following with pupils. In fact, viewers of Channel 4 voted Homer first place in 2001's 100 Greatest TV Characters and in 2003, he was made an honorary citizen of the Canadian city of Winnipeg! However, I must confess to sharing several attributes with Homer; in particular, his dreaming ability prior to attempting this piece of mathematics.

I am very fortunate that I have access to a Smart board, Internet and Autograph software. The lesson is introduced to the class with a video clip, that depicts Homer eating doughnuts in hell. This is followed by a discussion of the definition of a torus and a note in their jotters. Torus (plural tori) is the Latin word for cushion. Mathematically, it is a surface of revolution generated by revolving a circle in three dimensional space about an axis coplanar with the circle, which does not touch the circle. Examples of tori include inner tubes and the surfaces of doughnuts (or 'donuts' in the States), Homer's favourite snack.

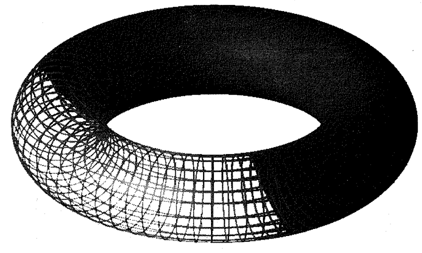
In groups, the following questions are posed:

Is a doughnut a polyhedron?

What is the volume of a doughnut?

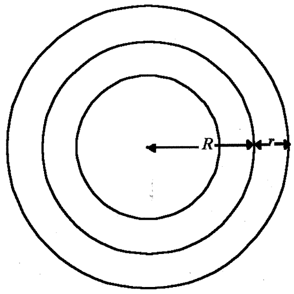
What is the surface area of a doughnut?

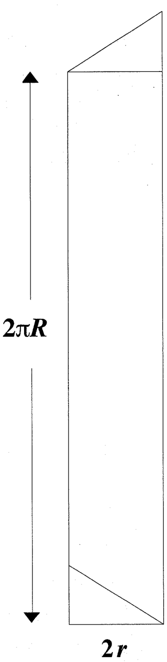
The answer to the first question is 'no' since a polyhedron is a geometric object with flat faces and straight edges. Answers to the second and third questions require a bit more reasoning!



Approach with S2-S4 pupils

With a good sketch and an active imagination, the idea here is to slice the doughnut through a circular cross-section and straighten it out so that it becomes a cylinder with a half-cylinder truncated from either end. It is then plausible to form a single cylinder from its object by mentally slicing off the half-cylinder from one end, turning it over, and replacing it at the other end so as to match the missing half-cylinder there.





The volume of our torus is much the same as the volume of a cylinder with the same radius r (and hence, base area πr^2) and height equal to the circumference of a circle of radius R.

Therefore the volume, V, of our torus is $V = \pi r^2 2\pi R = 2\pi^2 R r^2$.

In a similar way, the surface area of the torus equals the curved surface area of our cylinder. Slicing the cylinder open parallel to its axis and flattening it out, we form a rectangle whose height is the same as that of our cylinder and whose width is the circumference of its base. The area of this rectangle, and hence the surface area S of our torus is

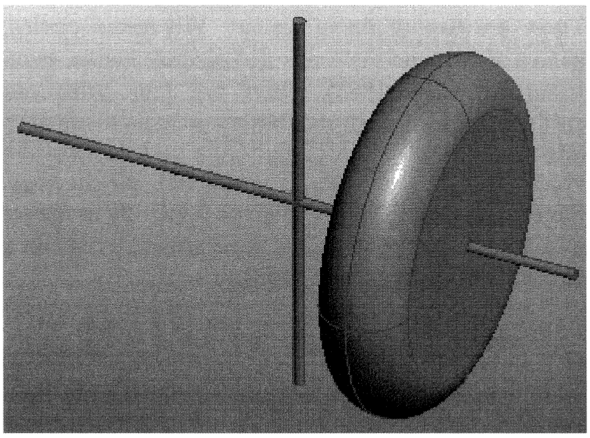
$$S = (2\pi R)(2\pi r) = 4\pi^2 R r.$$

Selected approach with Higher pupils

Acknowledgement: Both in this section and the next, I have leant heavily of the ideas and images of **Paul Bourke** of the University of Western Australia. My thanks to him for permission to employ them here. See Paul's solids of revolution at http://chuwm2.tripod.com/revolution.

I introduce the torus after delivering 'The Circle', when pupils are familiar with the equation of a circle $(x-a)^2 + (y-b)^2 = r^2$ with centre (a,b), radius r.

A torus is a hollow solid and its volume can thus be regarded as the difference of the volumes of the solids in the next two figures:



The larger solid (which corresponds to the outer surface of the torus) is obtained by revolving the upper half of the circle $(x-3)^2 + (y-3)^2 = 1$ around the x-axis.

Now, from the equation of the circle,

$$(y-3)^2 = 1 - (x-3)^2,$$

we can rearrange to give

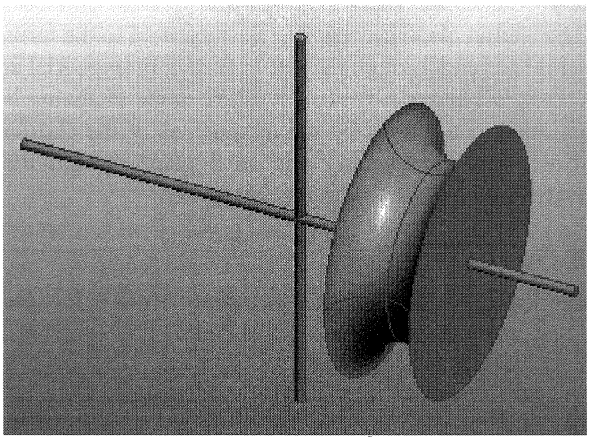
$$y = 3 \pm \sqrt{1-(x-3)^2}.$$

Of course, the upper part of the circle should correspond to

$$y = 3 + \sqrt{1-(x-3)^2}.$$

Therefore, the volume of the larger solid is given by

$$V = \pi \int_2^4 \left(3 + \sqrt{1-(x-3)^2}\right)^2 dx \qquad (1)$$



On the other hand, the smaller solid (which corresponds to the inner surface of the torus) is obtained by revolving the lower half of the circle $(x-3)^2 + (y-3)^2 = 1$ around the x-axis (above).

The lower part of the circle should correspond to

$$y = 3 - \sqrt{1-(x-3)^2}.$$

Therefore, the volume of the smaller solid is equal to

$$v = \pi \int_2^4 \left(3 - \sqrt{1-(x-3)^2}\right)^2 dx \qquad (2)$$

Combining (1) and (2), the volume of the torus is thus given by $V - v$ or

$$\pi \int_2^4 \left(3 + \sqrt{1-(x-3)^2}\right)^2 dx - \pi \int_2^4 \left(3 - \sqrt{1-(x-3)^2}\right)^2 dx$$

which can be further expressed as $2\pi^2 Rr^2$.

Approach with Advanced Higher pupils

The torus can be introduced after teaching parametric equations, for example:

$$x = r\cos t$$

$$y = r\sin t$$

which is one set of parametric equations for the circle.

A doughnut can be defined parametrically by

$$x(u,v) = (R + r\cos v)\cos u$$
$$y(u,v) = (R + r\cos v)\sin u$$
$$z(u,v) = r\sin v$$

where u, v are in the interval $[0, 2\pi)$, R is the distance from the centre of the tube to the centre of the torus, and r is the radius of the tube.

An equation in Cartesian coordinates for a torus radially symmetric about the z-axis is

$$\left(R - \sqrt{x^2+y^2}\right)^2 + z^2 = r^2,$$

and clearing the square root produces a quartic:

$$\left(x^2+y^2+z^2+R^2-r^2\right)^2 = 4R^2\left(x^2+y^2\right).$$

The surface area and volume of this doughnut are given by,

$$A = 4\pi^2 Rr = (2\pi r)(2\pi R)$$
$$V = 2\pi^2 Rr^2 = \left(\pi r^2\right)(2\pi R).$$

Using Autograph software (at least version 3.1), a torus can be examined using the following formula, in spherical polar equation mode:

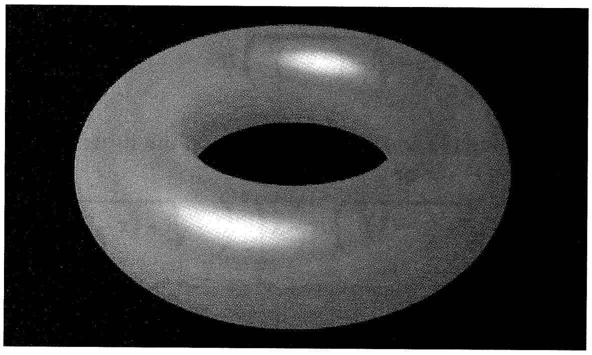
$$x = (c + a\cos\phi)\cos\theta$$
$$y = (c + a\cos\phi)\sin\theta$$
$$z = a\sin\phi.$$

Using plane $x + y + z = k$ will enhance the experience!

Challenge Question (Courtesy of Paul Bourke)

How many ways can a torus be cut (with a single plane) so that the resulting cross-sections are perfect circles?

I use r_1 = major radius = 2, r_2 = minor radius = 0.75:



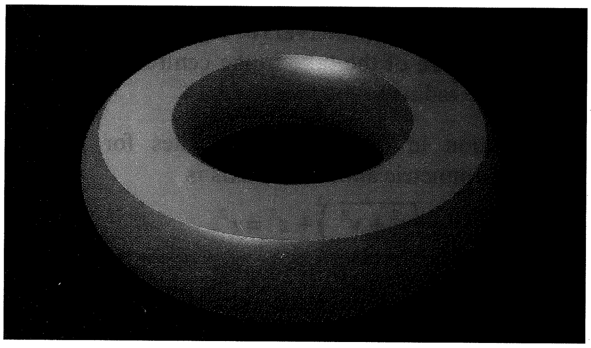
Answer: There are 3 ways!

Method 1

Horizontal slice resulting in two concentric circles
The radii of the two circles are given by

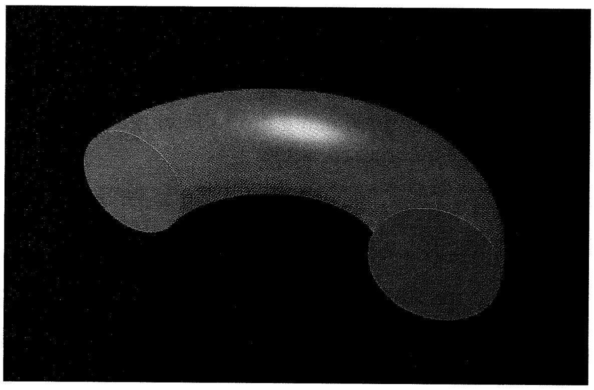
$$r_1 + \sqrt{r_2^{\,2} - h^2} \text{ and } r_1 - \sqrt{r_2^{\,2} - h^2}$$

where h is the distance of the cutting plane above the plane of the torus. Note that when the cutting plane is at a distance equal to the minor radius r_2, then there is only one solution. At greater distances, there are no solutions, as the plane doesn't cut the torus.



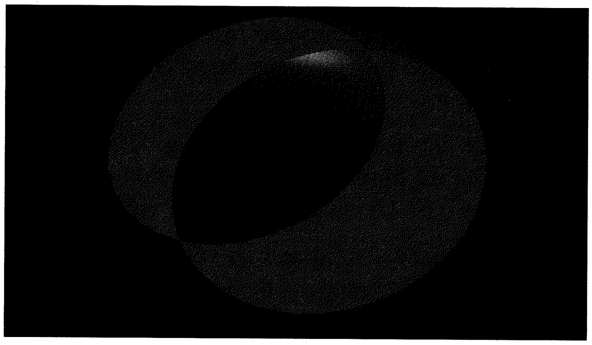
Method 2

Vertical slice resulting in two non-intersecting circles
Radius of circles = r_2.



Method 3

Angular cut resulting in two overlapping circles
Radius of circle = r_1.



These are two of the so called 'Villarceau' circles, named after French astromer and mathematician, Yvon Villarceau (1813-1883). There are four Villarceau circles passing through an arbitrary point on the surface of a torus.

For example, let the torus be given implicitly as the set of points on circles of radius three around points on a circle of radius five in the xy-plane

$$0 = \left(x^2 + y^2 + z^2 + 16\right)^2 - 100\left(x^2 + y^2\right).$$

Slicing with the $z = 0$ plane produces two concentric circles:

$$x^2 + y^2 = 2^2,$$
$$x^2 + y^2 = 8^2.$$

Slicing with the $x = 0$ plane produces two side-by-side circles:

$$(y-5)^2 + z^2 = 3^2,$$
$$(y+5)^2 + z^2 = 3^2.$$

Sample Villarceau circles can be produced by slicing with the plane $3x = 4z$. One is centred at (0, +3, 0), the other at (0, −3, 0). Both have $r = 5$.

They can be written in parametric form as

$$(x, y, z) = (4\cos\vartheta, +3 + 5\sin\vartheta, 3\cos\vartheta),$$
$$(x, y, z) = (4\cos\vartheta, -3 + 5\sin\vartheta, 3\cos\vartheta).$$

The slicing plane is chosen to be tangential to the torus while passing through its centre. Here it is tangential at $(16/5, 0, 12/5)$ and at $(-16/5, 0, -12/5)$. The angle of slicing is uniquely determined by the dimensions of the chosen torus, and rotating any one such plane around the vertical gives all of them for that torus.

References

1. Peter M. Higgins (1998) *Mathematics for the Curious*, Oxford University Press.
2. 'Volume of a torus' at http://chuwm2.tripod.com/revolution/torus2/index.htm