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ΕΤ ΕΥΤΕΛΕΙΑΣ
ΑΙ ΦΙΛΟΣΟΦΟΥ
ΜΕΝ ΑΝΕΥ
ΜΑΛΑΚΙΑΣ

VIII. *A* New Series *for the* Rectification *of the* Ellipsis ; *together with some* Observations *on the* Evolution *of the* Formula $(a^2 + b^2 - 2ab \cos\varphi)^n$. *By* James Ivory, A. M. *Communicated by* John Playfair, *Professor of Mathematics in the University of Edinburgh.*

[*Read Nov.* 7. 1796.]

Dear Sir,

HAVING, as you know, beftowed a good deal of time and attention on the ftudy of that part of phyfical aftronomy which relates to the mutual difturbances of the planets, I have, naturally, been led to confider the various methods of refolving the formula $(a^2 + b^2 - 2ab \cos\varphi)^n$ into infinite feries of the form $A + B \cos\varphi + C \cos 2\varphi + \&c.$ In the courfe of thefe inveftigations, a feries for the rectification of the ellipfis occurred to me, remarkable for its fimplicity, as well as its rapid convergency. As I believe it to be new, I fend it you, inclofed, together with fome remarks on the evolution of the formula juft mentioned, which, if you think proper, you may fubmit to the confideration of the Royal Society.

I am, Dear Sir,

Your's, &c.

Douglastown, near Forfar, }
20th October 1796. }

James Ivory.

To Mr John Playfair, Profeffor of Mathematics, &c.

LET s denote the excentricity of an ellipse, of which the semi-transverse axis is unity, and ϖ the length of the semicircle, radius being unity : Then,

if we put $e = \dfrac{1 - \sqrt{1 - s^2}}{1 + \sqrt{1 - s^2}}$,

half the periphery of the ellipsis will be

$$= \frac{\varpi}{1 + e}\left(1 + \frac{1^2}{2^2}e + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2}e^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2}e^6 + \frac{1^2 \cdot 1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}e^8 + \&c.\right),$$

the coefficients being the squares of the coefficients of the radical $\sqrt{1 - s^2}$.

THE common series is,

$$\varpi \times \left(1 - \frac{1 \cdot 1}{2 \cdot 2}s^2 - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4}s^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}s^6 - \&c.\right).$$

THE first of these series converges faster than the other on two accounts : first, because the coefficients decrease more rapidly; and, next, because e is very small in comparison of s, even when s is great : Thus, if s be $\frac{4}{5}$, e will be $\frac{1}{4}$, and $e^2 = \frac{1}{16}$.

IN order to point out the way in which the preceding series was discovered, let us suppose $(a^2 + b^2 - 2ab \cos\varphi)^n = A + B \cos\varphi + C \cos 2\varphi + \&c.$; and to determine the coefficients, A, B, C, &c. let us, with M. DE LA GRANGE, consider the quantity $(a^2 + b^2 - 2ab \cos\varphi)$ as the product of the two imaginary expressions $\left(a - bc^{\varphi\sqrt{-1}}\right)$, and $\left(a - bc^{-\varphi\sqrt{-1}}\right)$, where c denotes the number whose hyperbolic logarithm is unity. Then, by expanding the powers $\left(a - bc^{\varphi\sqrt{-1}}\right)^n$, and $\left(a - bc^{-\varphi\sqrt{-1}}\right)^n$ into the series $a^n\left(1 - a \cdot \frac{b}{a}c^{\varphi\sqrt{-1}} + \beta c^{2\varphi\sqrt{-1}} - \gamma c^{3\varphi\sqrt{-1}} + \&c.\right)$

and

instead of β & γ write $\beta\frac{b^2}{a^2}$ & $\gamma\frac{b^3}{a^3}$ & do the same in the first line of the next page. ———

and $a^n \Big(1 - \alpha . \frac{b}{a} c^{-\varphi\sqrt{-1}} + \beta c^{-2\varphi\sqrt{-1}} - \gamma c^{-3\varphi\sqrt{-1}} + \&c.$

we have $\alpha = n$, $\beta = \frac{n.\overline{n-1}}{1.2}$, $\gamma = \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$ &c.

Then multiplying thefe two feries together, and putting $2\cof m\varphi$ for its imaginary value $c^{+m\varphi\sqrt{-1}} + c^{-m\varphi\sqrt{-1}}$, we fhall find, on equating the terms,

$$A = a^{2n} \times \Big(1 + \alpha^2 . \frac{b^2}{a^2} + \beta^2 . \frac{b^4}{a^4} + \gamma^2 . \frac{b^6}{a^6} + \&c. \Big),$$

$$B = -2a^{2n} \times \Big(\alpha \cdot \frac{b}{a} + \alpha\beta \cdot \frac{b^3}{a^3} + \beta\gamma \cdot \frac{b^5}{a^5} + \&c. \Big),$$

and fo on.

Of the feveral feries for A, B, C, &c. the firft deferves particular attention, on account of the fimplicity of the law of its terms. It deferves the more attention, too, that the whole fluent

$\int \dot{\varphi} (a^2 + b^2 - 2ab \cof\varphi)^n$, generated while φ from 0 becomes $= \pi$, half the circumference of the circle, is $= A \pm \pi$: all the other terms of the fluent then vanifhing.

Suppose now, in an ellipfis, the femi-tranfverfe $= 1$, the excentricity $= \epsilon$, and φ an arch of the circumfcribing circle, reckoned from the extremity of the tranfverfe : then the fluxion of the correfpondent arch of the ellipfis, cut off by the fame ordinate, will be $= \dot{\varphi} \sqrt{1 - \epsilon^2 \cof^2\varphi}$.

In this expreffion, I write $\frac{1}{2} + \frac{1}{2} \cof 2\varphi$, for $\cof^2\varphi$: and put the refult, $\dot{\varphi} \sqrt{1 - \frac{\epsilon^2}{2} - \frac{\epsilon^2}{2} \cof 2\varphi} = \dot{\varphi} \sqrt{a^2 + b^2 - 2ab \cof 2\varphi}$, a and b being indeterminate quantities.

To determine a and b, we have $a^2 + b^2 = 1 - \frac{\epsilon^2}{2}$, and $2ab = \frac{\epsilon^2}{2}$:

whence $a + b = 1$, and $a - b = \sqrt{1 - \epsilon^2}$ fo that $a = \frac{1 + \sqrt{1 - \epsilon^2}}{2}$ and $b = \frac{1 - \sqrt{1 - \epsilon^2}}{2}$.

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I thus obtain $\dot\varphi \sqrt{1 - s^2 \cos^2\varphi} = \dot\varphi \sqrt{a^2 + b^2 - 2ab \cos 2\varphi}$: and, taking the whole fluent, while φ from 0 becomes $= \varpi$, it is manifest, from what has been premifed, that the femiperiphery of the ellipfis is $=$

$$\varpi \times a \times \left(1 + \frac{1}{2^2} \cdot \frac{b^2}{a^2} + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} \cdot \frac{b^4}{a^4} + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{b^6}{a^6} + \&c.\right),$$

or putting $\frac{b}{a} = e = \frac{1 - \sqrt{1 - s^2}}{1 + \sqrt{1 - s^2}}$, and $a = \frac{a}{a+b} = \frac{1}{1 + \frac{b}{a}} = \frac{1}{1 + e}$,

the femiperiphery of the ellipfis $= \frac{\varpi}{1 + e} \times$

$$\left(1 + \frac{1^2}{2^2} e^2 + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} e^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} e^6 + \&c.\right).$$

In this feries, as was before obferved, e is a fmall fraction even when s is very confiderable, and the coefficients are more fimple in the law of progreffion, and converge fafter, (efpecially in the firft terms), than in the common feries.

If we fuppofe the ellipfis to be infinitely flattened, in which cafe $s = 1$, and $e = 1$, and the femiperiphery $= 2$, this feries gives $2 = \frac{\varpi}{2} \times \left(1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} + \&c.\right)$, and fo

$$\frac{4}{\varpi} = 1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} + \&c.$$

But, we may remark, that as we have here obtained the fum of the fquares of the coefficients of the binomial when the exponent is $\frac{1}{2}$; fo, from the fame fource, we may determine the fum of the fquares of the coefficients correfponding to any other exponent, at leaft by a fluent.

For taking the whole fluent when $\varphi = \varpi$, we have

$$\int \left(a^2 + b^2 - 2ab \cos\varphi\right)^n \dot\varphi = a^{2n} \varpi \left(1 + \alpha^2 \cdot \frac{b^2}{a^2} + \beta^2 \cdot \frac{b^4}{a^4} + \gamma^2 \cdot \frac{b^6}{a^6} + \&c.\right)$$

and fo when $a = 1$, and $b = 1$,

$$\int \left(\frac{a^2 + b^2 - 2ab \cos\varphi}{\varpi}\right)^n \dot\varphi = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$$

Now,

Now, when $a = 1$, and $b = 1$, $\int \dot{\varphi} (a^2 + b^2 - 2ab \cos\varphi)^n = 2^n \times$

$\int \dot{\varphi} \left(\sin\frac{\varphi}{2}\right)^{2n}$, becaufe $2 \left(\sin\frac{\varphi}{2}\right)^2 = 1 - \cos\varphi$: we thus obtain

$$\frac{2^{2n} \times \int \dot{\varphi} \left(\sin\frac{\varphi}{2}\right)^{2n}}{\varpi} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$$

the whole fluent to be taken when $\varphi = \varpi$, or $\frac{\varphi}{2} = \frac{\varpi}{2}$.

If we put $x = \sin\frac{\varphi}{2}$, we fhall have

$$\frac{2^{2n} \times \int \frac{x^{2n} \dot{x}}{\sqrt{1 - x^2}}}{\frac{1}{2}\varpi} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c,$$

the whole fluent to be taken when $x = 1$; and in this formula n is any number fractional or integral, pofitive or negative; and α, β, γ, &c. the coefficients of the binomial raifed to a power of which the exponent is n.

When n is a whole pofitive number,

$\int \frac{x^{2n} \dot{x}}{\sqrt{1 - x^2}} = \frac{1 . 3 . 5 \dots (2n - 1)}{2 . 4 . 6 \dots 2n} . \frac{\varpi}{2}$, in the cafe when $x = 1$:

And fo, $2^{2n} \times \frac{1 . 3 . 5 \dots (2n - 1)}{2 . 4 . 6 \dots 2n} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$

Now, $2^{2n} \times \frac{1 . 3 . 5 \dots (2n - 1)}{2 . 4 . 6 \dots 2n}$ is no other than the coefficient of the middle term of a binomial, raifed to the power expreffed by $2n$: Hence we have a very curious property of thofe numbers: viz. *that the fum of the fquares of the coefficients of a binomial, the exponent being n, is equal to the coefficient of the middle term of a binomial, of which the exponent is 2n.*

Another remark, which I have to offer on this fubject, 'may be confidered not only as curious, in an analytical point of view, but as, in fome meafure, accomplifhing an object that has much engaged the attention of mathematicians.

In the computation of the planetary difturbances, it becomes

neceffary to evolve the fraction $(a^2 + b^2 - 2ab \cos\varphi)^{-\frac{1}{2}}$ into a

<div align="right">feries</div>

feries of this form, $A + B \cos \varphi + C \cos 2\varphi + $ &c. The quantities a and b reprefent the diftances of the difturbing planets from the fun; and when thefe bear fo great a proportion to one another, (as in the cafe of Jupiter and Saturn, or Venus and the Earth), that the fraction $\frac{b}{a}$ is large, it becomes extremely difficult to compute the coefficients A, B, &c. by feries, on account of the great number of terms that muft be taken in. This matter not a little perplexed the firft geometers who confidered this fubject, and they were obliged to approximate to the quantities fought by the method of quadratures, and by other artifices.

Two things are to be attended to with regard to the quantities A, B, C, &c. The firft is, That it is not neceffary to compute all of them feparately by feries, or by other methods: They form a recurring feries; and the two firft being fo computed, all the reft may be derived from them. The fecond thing is, That the quantities A and B having been computed for any exponent n, the correfpondent quantities are thence derived, by eafy formulæ, for the exponents $n + 1$, $n + 2$; $n - 1$, $n - 2$; and in general for the exponent $n + m$, m being any integer number, pofitive or negative.

From thefe remarks, it follows, that the whole difficulty lies in the computation of the two firft quantities, A and B; and that we are not confined to a given exponent n, but may choofe any one in the feries, $n + 1$, $n + 2$, &c.; $n - 1$, $n - 2$, &c.; that will render the computation moft eafy and expeditious.

Thus, in order to compute the quantities A and B, for the exponent $- \frac{3}{2}$, M. DE LA GRANGE makes choice of the exponent $+ \frac{1}{2}$, which, in the whole feries of exponents $+ \frac{3}{2}$, $+ \frac{1}{2}$, $- \frac{1}{2}$, $- \frac{3}{2}$, &c. is the moft favourable for computation, on account of the convergency of the coefficients of the feries for A and B.

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In confidering thefe fubjects, however, I have fallen upon a method of computing the quantities A and B for the exponent $-\frac{1}{2}$ by feries that converge fo faft, that, even taking the moft unfavourable cafe that occurs in the theory of the planets, two or three terms give the values required with a fufficient degree of exactnefs. This is what I am now to communicate.

We are then to confider the expreffion $(a^2 + b^2 - 2ab\cos\varphi)^{-\frac{1}{2}}$ $= \dfrac{1}{\sqrt{a^2 + b^2 - 2ab\cos\varphi}}$: for the fake of fimplicity in calculation, I write $\dfrac{b}{a} = c$, throwing out a altogether; and I fuppofe

$$\frac{1}{\sqrt{(1 + c^2 - c\cos\varphi)}} = A + B\cos\varphi + C\cos 2\varphi + \&c.$$

Let ψ be an angle, fo related to φ, that $\sin(\psi - \varphi) = c\sin\psi$: It is obvious, from this formula, that $\psi = \varphi$ when $\sin\psi = 0$, that is, when ψ is equal to 0, or to π, 2π, &c.

We have then, $\cos(\psi - \varphi) = \sqrt{1 - c^2\sin^2\psi}$: and taking the fluxions, $\dot\psi - \dot\varphi = \dfrac{c\cos\psi \times \dot\psi}{\cos(\psi - \psi)} = \dfrac{c\cos\psi \times \dot\psi}{\sqrt{(1 - c^2\sin^2\psi)}}$:

whence $\dot\varphi = \dot\psi \times \dfrac{\sqrt{1 - c^2\sin^2\psi} - c\cos\psi}{\sqrt{1 - c^2\sin^2\psi}}$

But $(\sqrt{1 - c^2\sin^2\psi} - c\cos\psi)^2 = 1 - c^2\sin^2\psi + c^2\cos^2\psi - 2c\cos\psi . \sqrt{1 - c^2\sin^2\psi} = 1 + c^2 - 2c^2\sin^2\psi - 2c\cos\psi \sqrt{1 - c^2\sin^2\psi}$, (becaufe $c^2\cos^2\psi = c^2 - c^2\sin^2\psi$) $= 1 + c^2 - 2c \times (c\sin\psi \times \sin\psi + \cos\psi \sqrt{1 - c^2\sin^2\psi})$. Now, if we write for $c\sin\psi$ its equal, $\sin(\psi - \varphi)$, and for $\sqrt{1 - c^2\sin^2\psi}$ its equal, $\cos(\psi - \varphi)$, we fhall have $c\sin\psi \times \sin\psi + \cos\psi \times \sqrt{1 - c^2\sin^2\psi} = \sin(\psi - \varphi) \times \sin\psi + \cos\psi \times \cos(\psi - \varphi) = \cos\varphi$: which being fubftituted, there comes out $(\sqrt{1 - c^2\sin^2\psi} - c\cos\psi)^2 = 1 + c^2 - 2c\cos\varphi$.

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OUR fluxional formula thus becomes $\dot{\varphi} =$ $\dot{\psi} \dfrac{\sqrt{1 + c^2 - 2c \cos \varphi}}{\sqrt{1 - c^2 \sin^2 \psi}}$: whence $\dfrac{\dot{\varphi}}{\sqrt{1 + c^2 - 2c \cos \varphi}} = \dfrac{\dot{\psi}}{\sqrt{(1 - c^2 \sin^2 \psi)}}$.

I NEXT transform the quantity $\sqrt{1 - c^2 \sin^2 \psi}$ as in the inveſtigation for the elliptic ſeries, and putting $c' = \dfrac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}$, I find $\sqrt{1 - c^2 \sin^2 \psi} = \dfrac{\sqrt{1 + c'^2 + 2c' \cos 2\psi}}{1 + c'}$, and ſo $\dfrac{\dot{\varphi}}{\sqrt{1 + c^2 - 2c \cos \varphi}} = \dfrac{(1 + c') \dot{\psi}}{\sqrt{1 + c'^2 + 2c' \cos 2\psi}}$.

Now, taking the fluents when $\varphi = \pi$, and $\psi = \pi$, we ſhall have $\int \dfrac{\dot{\varphi}}{\sqrt{1 + c^2 - 2c \cos \varphi}} = A \times \pi$: And according to the method of M. DE LA GRANGE, $\int \dfrac{\dot{\psi}}{\sqrt{1 + c'^2 + 2c' \cos 2\psi}} = \pi \times$ $\left(1 + \dfrac{1^2}{2^2} c'^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \&c. \right)$: Hence $A = (1 + c') \times$ $\left(1 + \dfrac{1^2}{2^2} c'^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \&c. \right)$. And in this value of A, c' will be a ſmall fraction, even though c be large; and the ſeries will therefore converge very faſt.

BUT, taking the value of A directly in a ſeries, we have $A = 1 + \dfrac{1^2}{2^2} c^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c^4 + \&c.$ And ſo $1 + \dfrac{1^2}{2^2} c^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c^4 + \&c. = (1 + c') \times \left(1 + \dfrac{1^2}{2^2} c'^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \&c. \right)$. Now, the two ſeries being exactly alike, it is evident that we may transform the one, as we have transformed the other, and that, if we put $c'' = \dfrac{1 - \sqrt{1 - c'^2}}{1 + \sqrt{1 - c'^2}}$ we ſhall have $1 + \dfrac{1^2}{2^2} c'^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c'^4 = (1 + c'') \times \left(1 + \dfrac{1^2}{2^2} c''^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c''^4 + \&c. \right)$: whence $A = (1 + c')$ $(1 + c'') \left(1 + \dfrac{1^2}{2^2} c''^2 + \dfrac{1^2 . 3^2}{2^2 . 4^2} c''^4 + \&c. \right)$.

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It is manifeft we may proceed in this manner as far as we pleafe, and that, if we put $c'''' = \frac{1 - \sqrt{1 - c''^2}}{1 + \sqrt{1 - c''^2}}$; $c''''' = \frac{1 - \sqrt{1 - c'''^2}}{1 + \sqrt{1 - c'''^2}}$ and fo on, we fhall have the value of A in an infinite product, $A = (1 + c') \times (1 + c'') \times (1 + c''') (1 + c'''') \times$ &c, the quantities c', c'', c''', c'''', &c. converging very rapidly.

Nothing more feems to be wifhed for, with regard to the computation of the quantity A: fince we can, by methods fufficiently fimple, exhibit the value of it in feries that fhall converge as faft as we pleafe. By a fimilar mode of reafoning, I find the feries $1 - \frac{1}{2^2}\gamma^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\gamma^4 - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\gamma^6 +$ &c. (which occurs in determining the time of a body's defcent in the arch of a circle), $= (1 - c) \times \left(1 + \frac{1^2}{2^2}c^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}c^6 +\right.$ &c.) where $c = \frac{\sqrt{1 + \gamma^2} - 1}{\sqrt{1 + \gamma^2} + 1}$: fo that the fummation of this feries alfo is accomplifhed by the method above.

I have now only to explain the method of computing B. For this purpofe I refume,

$$\frac{1}{\sqrt{1 + c^2 - 2c\cos\varphi}} = A + B\cos\varphi + C\cos 2\varphi + \&c.$$

Multiply by $2\cos\varphi$, and there refults

$$\frac{2\cos\varphi}{\sqrt{1 + c^2 - 2c\cos\varphi}} = B + (2A + C)\cos\varphi + \&c.$$

whence it is manifeft that the whole fluent

$$\int \frac{2\cos\varphi \times \dot\varphi}{\sqrt{1 + c^2 - 2c\cos\varphi}},$$ when $\varphi = \pi$, is equal to $B \times \pi$.

From the preceding inveftigation we have $\frac{\dot\varphi}{\sqrt{1 + c^2 - 2c\cos\varphi}} = \frac{\dot\psi}{\sqrt{1 - c^2 \sin^2\psi}}$, and $\cos\varphi = c\sin^2\psi + \cos\psi\sqrt{1 - c^2\sin^2\psi}$,

Z. whence

whence $\dfrac{2\dot\phi\cos\phi}{\sqrt{1+c^2-2c\cos\phi}} = \dfrac{2c\dot\psi\sin^2\psi}{\sqrt{1-c^2\sin^2\psi}} + 2\dot\psi\cos\psi.$ **Again,**

$2\sin^2\psi = 1 - \cos 2\psi$, and $\dfrac{1}{\sqrt{1-c^2\sin^2\psi}} = \dfrac{(1+c')}{\sqrt{1+c'^2+2c'\cos 2\psi}}$, c'

being $= \dfrac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}$: these substitutions being made, we get

$\dfrac{2\dot\phi\cos\phi}{\sqrt{1+c^2-2c\cos\phi}} = c \times \dfrac{(1+c')\dot\psi}{\sqrt{1+c'^2+2c'\cos 2\psi}} - c \times \dfrac{(1+c')\dot\psi\cos 2\psi}{\sqrt{1+c'^2+2c'\cos 2\psi}}$

$+ 2\dot\psi\cos\psi.$

SUPPOSE now, $\dfrac{1}{\sqrt{1+c'^2+2c'\cos 2\psi}} = A' - B'\cos 2\psi + c'\cos 4\psi - \&c.$

it is evident, from what goes before, that, taking the fluents of the above fluxions, when ϕ and $\psi = \varpi$, we shall have $B \times \varpi$

$= c \times (1+c') \times \left(A' + \dfrac{B'}{2}\right) \times \varpi$, and so $B = c \times (1+c') \times$

$\left(A' + \dfrac{B'}{2}\right).$

THE values of A' and B', in series according to the method of M. DE LA GRANGE, are

$$A' = 1 + \frac{1^2}{2^2}c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c'^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}c'^6 + \&c.$$

$$\tfrac{1}{2}B' = \left(\frac{1}{2}c' + \frac{1}{2}\cdot\frac{1\cdot 3}{2\cdot 4}c'^3 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}c'^5 + \&c.\right)$$

which series converge very fast, on account of the smallness of c' in respect of c.

IF, however, it be required to find the value of B by series still more converging, we may easily do so: For it is manifest that B and B' are similar functions of c and c': and that if we

make $c'' = \dfrac{1-\sqrt{1-c'^2}}{1+\sqrt{1-c'^2}}$, $c''' = \dfrac{1-\sqrt{1-c''^2}}{1+\sqrt{1-c''^2}}$, and so on, and put

A'',

A″, A‴, &c. ; B″, B‴, &c. to denote the correſponding values of A′ and B′, we ſhall have

$$B = c \cdot (1 + c')\left(A' + \frac{B'}{2}\right)$$

$$B' = c' \cdot (1 + c'')\left(A'' + \frac{B''}{2}\right)$$

$$B'' = c'' \cdot (1 + c''')\left(A''' + \frac{B'''}{2}\right) \&c.:$$

Now, remarking that $A' = (1 + c'') A''$; $A'' = (1 + c''') A'''$, &c. we have the following values of B :

$$B = c \times \left(1 + \frac{c'}{2}\right) \cdot (1 + c') \cdot (1 + c'') A'' + \frac{c}{2} \cdot \frac{c'}{2}(1 + c')(1 + c'') B''.$$

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2}\right)(1 + c')(1 + c'')(1 + c''') A''' + \frac{c}{2} \cdot \frac{c'}{2} \cdot \frac{c''}{2}(1 + c')$$

$$(1 + c'')(1 + c''') B'''.$$

And we may proceed in this manner to find the value of B in ſeries that ſhall converge as faſt as we pleaſe.

As the quantities c', c'', c''', &c. diminiſh very faſt, the ſeries A′, A″, A‴ will approach rapidly to unity, and B′, B″, B‴ will decreaſe rapidly to nothing : Hence we have ultimately,

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c.\right) \times (1 + c')(1 + c'')(1 + c''')$$

&c.

or, ſince $A = (1 + c')(1 + c'')(1 + c''')$ &c.:

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c.\right) \times A.$$

We ſhall beſt ſee the degree of convergency of the quantities c, c', c'', &c. if we take the infinite ſeries by which they are derived one from another. Now, if $y = \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}}$, then alſo $y =$

$\frac{x^2}{4} + \frac{x^4}{8} + \frac{5x^6}{64} + \frac{7x^8}{128} + \&c.$: whence it is obvious, that in the

ſeries of quantities c, c', c'', &c. the fourth part of the ſquare of

any

any term is nearly equal to the following term, and the rapidity with which the feries decreafes is therefore very great.

THE method, then, that refults from the preceding inveftigations for computing A and B, is fhortly this :

Put $c' = \dfrac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}$: and compute

$$1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} c'^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 5^2 \cdot 6^2} c'^6 + \&c. = M,$$

and $\dfrac{1}{2} c' + \dfrac{1}{2} \cdot \dfrac{1 \cdot 3}{2 \cdot 4} c'^3 + \dfrac{1 \cdot 3}{2 \cdot 4} \cdot \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} c'^5 + \&c. = N.$

Then $A = (1 + c') \times M$, and

$$B = c \times (1 + c') \times (M + N).$$

THE feries M and N will converge fo faft, even in the moft unfavourable cafe that occurs in the theory of the planets, that the firft three terms will give the fums fufficiently exact ; and it will therefore not be neceffary to have recourfe to the more converging feries A" and B".

SUCH is the method that I had firft imagined, for facilitating thefe fort of computations. I have fince found, however, that by means of the common tables of fines and tangents, the quantities A and B may be computed in a ftill eafier way from the expreffions,

$$A = (1 + c') (1 + c'') (1 + c''') \&c.$$

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} \cdot \frac{c''}{2} + \frac{c'}{2} \cdot \frac{c''}{2} \cdot \frac{c'''}{2} + \&c.\right) \times A.$$

FOR if $c = \text{fin } m$, then $\sqrt{1 - c^2} = \text{cof } m$ and $c' = \dfrac{1 - \text{cof } m}{1 + \text{cof } m}$

$= \tan^2 \dfrac{m}{2}$: confequently $1 + c' = \text{fec}^2 \dfrac{m}{2}$. In like manner, if

$c' = \text{fin } m'$, $c'' = \text{fin } m''$, &c. we fhall have $\text{fin } m' = \tan^2 \dfrac{m}{2}$;

fin m'

fin $m''=$ tan^2 $\frac{m'}{2}$, and fo on: And $1+c''=$ fec^2 $\frac{m'}{2}$; $1+c'''=$

fec^2 $\frac{m''}{2}$, and fo on. Thus:

$$A = \text{fec}^2\,\frac{m}{2} \times \text{fec}^2\,\frac{m'}{2} \times \text{fec}^2\,\frac{m''}{2} \times \&c.$$

To find the logarithm of A, we have then only to add toge-
ther the logarithm fecants of the angles $\frac{m}{2}$, $\frac{m'}{2}$, $\frac{m''}{2}$, &c. to dimi-
nifh the fum by as many times the radius as there are fecants,
and to take twice the remainder. As the angles m, m', m'', &c.
decreafe very faft, it will feldom be neceffary to compute more
than two or three of them.

THE feries $\left(1 + \frac{c'}{2} + \frac{c'}{2}\cdot\frac{c''}{2} + \frac{c'}{2}\cdot\frac{c''}{2}\cdot\frac{c'''}{2} + \&c.\right)$ is alfo readily com-
puted from the tables; becaufe the logarithms of c', c'', c''', &c. be-
ing the fines of the angles m', m'', &c. are all found in the ta-
bles.

As an example, let $c = 0.72333$: which is the fraction that
arifes from dividing the mean diftance of Venus from the fun,
by the mean diftance of the Earth; and this is the moft unfa-
vourable cafe that occurs in the theory of the planets: Then to
compute A, I find, in the table of natural fines, that 0.72333 cor-
refponds to $46°\ 19'\ 48\frac{1}{3}''$: we have therefore

$$m = 46°\ 19'\ 48\frac{1}{3}''$$

L. tan $\frac{m}{2}=$ l. tan $23°\ 9'\ 54\frac{1}{6}'' = 9.6313206$ | L. fec $\frac{m}{2} = 10.0365070$

$$2$$
$$\overline{}$$

L. fin $m' = \quad 9.2626412$

L. $m' = 10°\ 32'\ 57'$

L. tan $\frac{m'}{2}=$ l. tan $5°\ 16'\ 28\frac{1}{2}'' = 8.9652949$ | L. fec $\frac{m'}{2} = 10.0018429$

$$2$$
$$\overline{}$$

L. fin $m'' = \quad 7.9305898$

L. $m'' = 0°\ 29'\ 18''$

L. tan

L. tan $\frac{m''}{2}$ = l. tan 0° 14′ 39″ = 7.6295664 | L. sec $\frac{m''}{2}$ = 10.0000039

$$\begin{array}{r} 2 \\ \hline \end{array}$$

L. sin m''' = 5.2591328 | 0.0383538

| 2

As m''' will only be a few seconds, it may be neglected. | L. A = 0.0767076

Hence A = 1.19318

To compute B, let S = $1 + \frac{c'}{2} + \frac{c'}{2}\cdot\frac{c''}{2}$ + &c.

1. = 1.000000

L. c' = l. sin m' = 9.2626412 ; | $\frac{c'}{2}$ = 0.091540

L. c'' = l. sin m'' = 7.9305898 | $\frac{c'}{2}\cdot\frac{c''}{2}$ = 0.000390

l. c'. c'' = 7.1932310 | S = 1.091830

B = $c \times S \times A$.

L. c = 1.8593365

L. S = 0.0381948

L. A = 0.0767076

L. B = 1.9742389, and B = 0.942408