EQUIVALENT OF
ELLIPTIC INTEGRALS

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Abstract

The finite elliptic similar integrals of second kind are well known as

$$L = a.\int_0^1 (1+(b/a)^2.(k^r/(1-k^r))^{((2.r-2)/r)})^{(1/2)}.dk$$

Those integrals cannot be solved by any classical method. In this paper, we prove that the above equation can be replaced by

$$L = a.(1+(b/a)^s)^{1/s}$$

As it is well known, on the positive Cartesian, all astroids are expressed by:
$(x/a)^r + (y/b)^r = 1$

where a, b, and r are any positive constant real numbers.

Using this equivalency and when (r=2) the perimeter of an ellipse is estimated at full-range with a maximum error %=-0,000002432

Full-range is (1<b/a<infinity).Ram.

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Application

The aim of this work is to find the most accurate estimation for the total arc length of the astroids on the positive Cartesian, mainly for the ellipses.

To estimate the perimeter of an ellipse, there are thousand of formulas:
Kepler, Euler, Muir, Ramanujan, …...many mathematicians have tried to give an accurate approximation for the perimeter of an ellipse, but only for the ellipse!
Here, we will prove a NEW EXACT formula applicable to all the astroids, ellipse included.
We will propose a very accurate, approximate solution of this formula.
This proposition was declared on San Francisco IAENG conference in 2008 without giving the proof.
Here, the proof will be introduced the first time.
The math world has never seen such an accurate estimation.
World recorded error is %=0.00145..
New record error is %=-0.000002432…

Introduction to the ELLIPTIC SIMILAR FINITE INTEGRALS

$(x/a)^r + (y/b)^r = 1$ (1i)

astroid family is considered.
We search for the total arc length (**L**) on the positive Cartesian

$x = k.a$ is written, then from (1i)
$y = b.(1-k^r)^{(1/r)}$ is found

We differentiate these expressions

$dx^2 = a^2.dk^2$
$dy^2 = b^2.(1-k^r)^{((2-2.r)/r)} k^{(2.r-2)}.dk^2$ then
$dL^2 = dx^2 + dy^2$ (2i)
is considered
$dL^2 = (a^2 + b^2.(1-k^r)^{((2-2.r)/r)} k^{(2.r-2)}).dk^2$
is written and

$$L = a.\int_0^1 (1+(b/a)^2.(k^r/(1-k^r))^{((2.r-2)/r)})^{(1/2)}.dk$$

is found (3i)

Example: r=1
(we substitute r=1 in (3i))

$$L=a.\int_0^1 (1+(b/a)^2.(k^1/(1-k^1))^{((2.1-2)/1)})^{(1/2)}.dk$$

$$L=a.\int_0^1 (1+(b/a)^2.(...)^0)^{1/2}.dk$$

$$L=a/a.(a^2+b^2)^{(1/2)}.\int_0^1 dk$$

or, with linear writing

L=(a^2+b^2)^(1/2)   is found

Example: r=2/3
(we substitute r=2/3) in (3i))
No solution. But when a=b

L=a*int[0 to 1] (1+((1-k^(2/3)/k^(2/3)))^(1/2)*dk

L=a*int[0 to 1] k^(-1/3)*dk
L=a*3/2   is found

Example: r=2
(we substitute r=2 in (3i))
No solution. (Only special series terms solution) and when a=b,

L=a*int[0 to 1]((1+k^2/(1-k^2))^(1/2)*dk

L=a*(Pi/2)   is found by **definition.**

ELLIPTIC SIMILAR FINITE INTEGRALS ARE EXPRESSED BY

$$L=a.\int_0^1 (1+(b/a)^2.(k^r/(1-k^r))^{((2.r-2)/r)})^{(1/2)}.dk$$

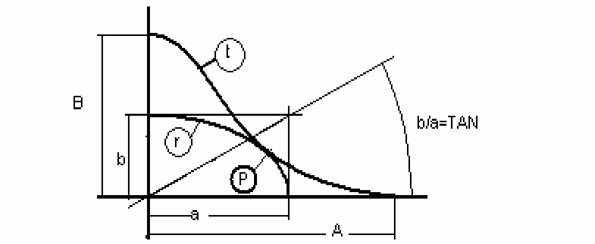
When a=1, the unit total arc length of the astroid, on the positive Cartesian, is evaluated.
We will prove: say (b/a=TAN)
$L1=(1+TAN^s)^{(1/s)}$
or, with linear writing

L1=(1+TAN^s)^(1/s)   (4i)

Only, for (r=2, the ellipse) an eccentricity is defined.[e]
For b<a   e=(1-TAN^2)^(1/2)
For b>a   e=(1-1/TAN^2)^(1/2)
For other astroids (e) has not a signification.
So, we will not use [e], but [TAN]. Valid for (0<r<infinite)

PROOF

To reach to the formula L1^s=1+TAN^s



The astroid family

(x/a)^r+(y/b)^r=1   (1)
where r=Constant, is enveloped by
(x/A)^t+(y/B)^t=1   where t=t(x)   (2)

We search for a relation **f(a,b,r,A,B,t)=0**

At the touching point (P) of the graphs we write   -the slopes are equal
          -the coordinates are equal

For the coordinates we write
y=b/a*(a^r-x^r)^(1/r)=B/A*(A^t-x^t)^(1/t)
(3)
For the slope of the enveloped astroid we write
dy/dx=-(b/a)^r*(x/y)^(r-1)
dy/dx=-(b/a)*((x^r)/(a^r-x^r))^((r-1)/r)
(4)
For the slope of the envelope itself
-say (x/A)^t=U ; (y/B)^t=V then,

U+V=1   (5)
dU+dV=0   (6)

we have
t*ln(x/A)=lnU
t*ln(y/B)=lnV (7)

When we differentiate (7), we write

dt*ln(x/A)+t*(dx/x)=dU/U
dt*ln(y/B)+t*(dy/y)=dV/V (8)

and there from,

dU=U*(dt*ln(x/A)+t*(dx/x))
dV=V*(dt*ln(y/B)+t*(dy/y)) (9)

Considering (7), the expression (6) is written as

U*(dt*1/t*lnU+t*dx/x)+V*(dt*1/t*lnV+t*dy/y)=0 (10)

and there from

V*t*dy/y=-U*(dt/t*lnU+t*dx/x)-V*(dt/t*lnV) (11)

dy/dx=-y/(V*t)*U*((dt/dx*1/t*lnU+t/x))+V*dt/dx*1/t*lnV) (12)

taking (U and t/x) out of the parenthesis

dy/dx=-U/V*y/x*(1+dt/dx*x/t^2*1/U*(U*lnU+V*lnV)) is written (13)

say

N=(1+dt/dx*x/t^2*1/U*(U*lnU+V*lnV)) (14)

dy/dx=-U/V*y/x*N is written (15)

For the equality of the slopes, we write (15)=(4).

Considering also (3), we write

(b/a)*((x^r)/(a^r-x^r))^((r-1)/r)=U/V*1/x*(B/A)*(A^t-x^t)^(1/t)*N (16)

Replacing (4) and (5) in (16) we write

(b/a)^r*(x/y)^(r-1)=(B/A)^t*(x/y)^(t-1)*N (17)

say (B/A)=E (18)

Use (3), we write (17) as follows

(b/a)^r=E^t*(x/(E*(A^t-x^t)^(1/t)))^(t-r)*N (19)
and there from

b^r=E^r*a^r*(x^t/(A^t-x^t))^((t-r)/t)*N
is written (20)

using (3) and (20), the expression (1) is written as follows

(x/a)^r+((A^t-x^t)^(r/t)*(A^t-x^t)^((t-r)/t))/a^r*x^(t-r)*N=1 (21)

x^t*N+A^t-x^t=a^r*x^(t-r)*N (22)

A^t=a^r*x^(t-r)*N-x^t*(N-1) (23)

then ,from (23) we get

x=((A^t-x^t*(1-N))/a^r*N)^(1/(t-r)) (24)

a^r*x^(t-r)*N=A^t-x^t*(1-N) (25)

using (25) in (20)

b^r=E^r*(A^t-x^t*(1-N))/(A^t-x^t)^(t-r)/t)
A^t=(b/E)^r*(A^t-x^t)^(t-r)/t)+x^t*(1-N)
is written (26)

(26)=(23) then,

(a/b*E)^r*x^(t-r)*N=(A^t-x^t)^((t-r)/t)
is written (27)

A^t-x^t=(a/b*E)^(r*t/(t-r))*x^t*N^(t/(t-r))
is written (28)

A^t=x^t*(1+((a/b)*E)^((r*t)/(t-r))*N^(t/(t-r))) is written (29)

From (29) we get (x)

x=A/(1+((a/b)*E)^((r*t)/(t-r))*N^(t/(t-r)))^(1/t) (30)

(30)=(24) then,

((A^t-x^t*(1-N))/(a^r*N))^(1/(t-r))=A/(1+((a/b)*E)^((r*t)/(t-r))*N^(t/(t-r)))^(1/t) (31)

(A^t-x^t*(1-N))/a^r*N=(A^(t-r)*b^r)/(b^(r*t/(t-r))+(a*E)^(r*t)/(t-r))*N^(t/(t-r)))^((t-r)/t) (32)

We take ((t-r)/r*t) power of both sides, We proceed, then we take the power (r*t/(r-t)) of both sides

say r*t/(r-t)=s (33)
we write

(A^t-x^t*(1-N))^(s/r)*A^t=a^s*N^(s/r)+(b/E)^s (34)

For the astroids of the same power, when b/a=TAN=Constant dt/dx=dt/dTAN*dTAN/dx=0 and **N=1**
then, (35)

A^(t*s/r)*A^t=A^s=a^s+(b/E)^s
is written (36)

that is:

(a/A)^s+(b/B)^s=1 (37)

In a symmetric case, when (A=B=K) , we write

K^s=a^s+b^s (38)

Say (K/a=L1) ; (b/a=TAN)

L1^s=1+TAN^s is written (38)L

L1=(1+TAN^s)^(1/s) **is proven**

Cracking

In this section we study the total arc length. The reasoning (38)L means,

((L1)1)^s1=1+TAN1^s1
((L1)2)^s2=1+TAN2^s2
…..
((L1)n)^sn=1+TANn^sn

This expression is implicit!

To crack this implicit expression, first we get the real data of (s).We know (L1). TAN=b/a is given. (s) is found.

The evaluation of (L1) is done by summing a couple millions segments of dL1.We suppose; *we have no idea about integrals. They are unsolvable practically!*

Application : (case r=2 the ellipse)

An application was posted inWCECS2008. This is an update, an expended version.

(x/a)^r+(y/b)^r=1 (1)
is considered

dL1=(dx^2+dy^2)^(1/2) are summed and

L1=Sum[dL1] is obtained

(1+TAN^s-L1^s=0) gives an [sReal] graph as shown in Figure.1(case r=2; the ellipse)

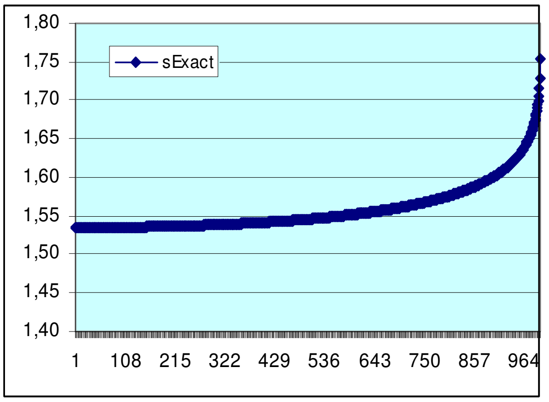


Fig.1 sExact for ellipse

We start the cracking:
This graph looks like an astroid.We write a math Model expression for this similarity:

sMod=d1+b1*(1-((x-c1)/a1)^p)^(1/p)+(F+m1*x^v1+n1*x^w1)

We overlap sExact & sMod graphs using the following parameters.Fig.2 shows the overlapping.

parameters	values
a1	1000
(sm-sM)=b1	-0,193967895182134
c1	0
d1	0,000000000000000
p	***2,980000000000000***
sM=F	1,728896430843500
m1	0,000000000000000
v1	1
n1	0
w1	1,00

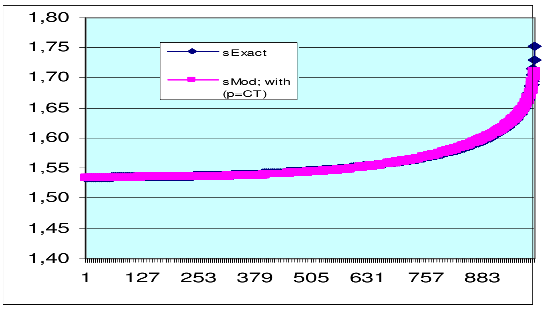


Fig.2 Overlapping of sExact&sMod

The overlapping is not good because we used [p=Ct].It should be variable. Then an exact fitting can be realized. So, we write

Error of the overlapping=0. For this we say (sMod-sExact)/sMod=0

This cracks [p=Constant] and gives a new graph for [p].Fig.3 shows [pExact] graph

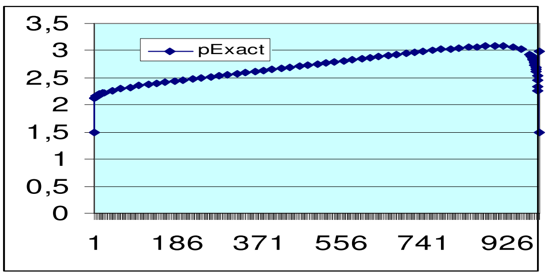


Fig.3 pExact giving error %=0

But this is nothing, than being a graph. We must write a math model for this graph.

We continue cracking. We say it looks like an astroid with the following parameters

pMod=d2+b2*(1-((x-c2)/a2)^q)^(1/q)+(G+m2*x^v2+n2*x^w2)

parameters	values
a2	500
b2	0,3
c2	500
d2	0,000
q	**6**
G	**1,965**
m2	**0,000996000**
v2	1
n2	0
w2	1

Fig.4 shows the overlapping of pExact&pMod graphs

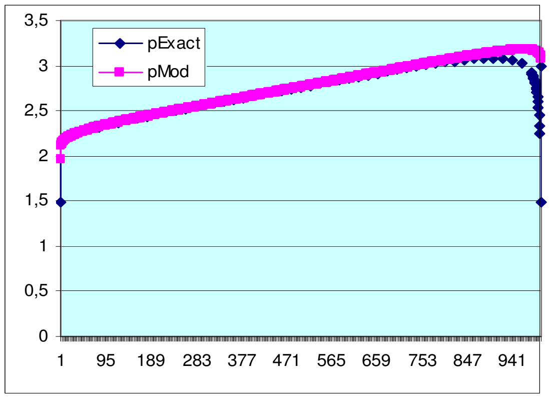


Fig.4 pMod do not fits pExact accurately

We say the parameters values estimated for a good overlapping was not correct. So, again, we write

Error of the overlapping=0.For this we say (pMod-pExact)/pMod=0

We attack the parameters of pMod.
We chose (b2; m2; G) consecutively

b2Mod ;m2Mod; GMod and the values of the parameters, and the corresponding new pMod , new sMod and also their overlapping graphs are as follows:

What we are doing is to correct pMod.

b2Mod=d3+b3*(1-((x-c3)/a3)^r)^(1/r)+(H+m3*x^v3+n3*x^w3)

pMod=d2+b2Mod*(1-((x-c2)/a2)^q)^(1/q)+(G+m2*x^v2+n2*x^w2)

sMod=d1+b1*(1-((x-c1)/a1)^pMod)^(1/pMod)+(F+m1*x^v1+n1*x^w1)

parameters	values
a3	500
b3	0,340000
c3	500
d3	0,000000
r	23
H	0,638500
m3	0,000040
v3	1,000000
n3	-0,000001
w3	1,600000

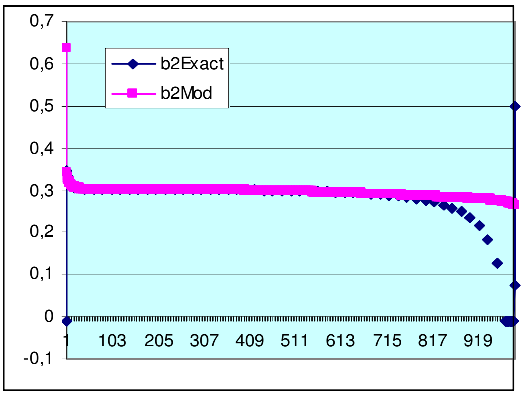


Fig.5 The overlapping is not good

So, we continue the correction, now with m2Mod

m2Mod=d4+b4*(1-((x-c4)/a4)^t)^(1/t)+(J+m4*x^v4+n4*x^w4)

pMod=d2+b2Mod*(1-((x-c2)/a2)^q)^(1/q)+(G+m2Mod*x^v2+n2*x^w2)

sMod=d1+b1*(1-((x-c1)/a1)^pMod)^(1/pMod)+(F+m1*x^v1+n1*x^w1)

parameters	values
a4	500
b4	-0,000906
c4	500
d4	0
t	12
J	0,000092
m4	0,0000000000
v4	1
n4	0
w4	1

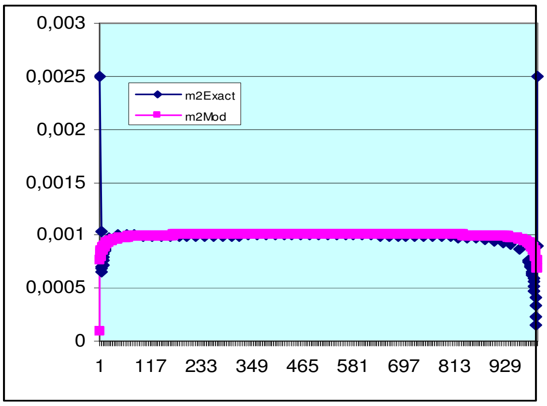


Fig.6 The overlapping is not good

So, we continue the correction, now with GMod

GMod=d5+b5*(1-((x-c5)/a5)^u)^(1/u)+(K+m5*x^v5+n5*x^w5)

pMod=d2+b2Mod*(1-((x-c2)/a2)^q)^(1/q)+(GMod+m2Mod*x^v2+n2*x^w2)

sMod=d1+b1*(1-((x-c1)/a1)^pMod)^(1/pMod)+(F+m1*x^v1+n1*x^w1)

parameters	values
a5	1000
b5	-0,797500000
c5	0
d5	0
u	8,37
K	1,171710000000000
m5	0,000001000000000
v5	0
n5	-0,0000011
w5	1,4

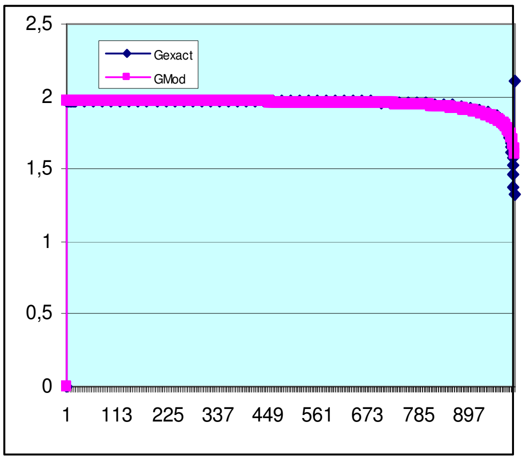


Fig.7 The overlapping looks good.

We say, these 5 stages evaluations are sufficient for an accurate **continuous estimation** of the total arc length on the positive Cartesian. We stop there.

We may continue !

We will control the error % for the whole range (1<b/a=TAN<infinity). We write:

Error %=(Lestimated-LExact)/Lestimated

Fig.8 shows the final error % graph.
Overall max.error %=-0,000002432900942

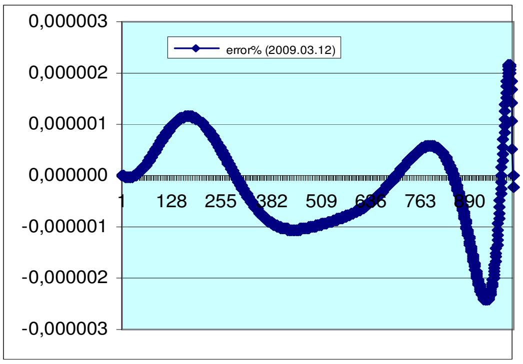


Fig.8 final error % graph (1<TAN<infini)

This accuracy is to be compared with the world known error %=0,00145….

Fig.9 is a comparison graph for the estimations of Master Ramanujan (dead 1920) with the estimation of Necat for (1<TAN<10)

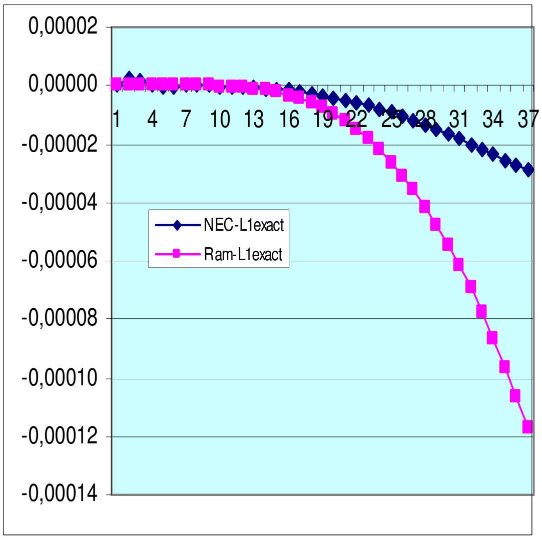


Fig.9 comparison Ram&Nec for dL1

Ramanujan is Master when TAN<2,5

A numerical example

$$L = a.\int_0^1 (1+(b/a)^2.(k^r/(1-k^r))^{((2.r-2)/r)})^{(1/2)}.dk$$

solve the above integral for (a=1 ;b=5;r=2)
No solution, except mathematical tools.
We can solve L=a*(1+TAN^s)^(1/s) with an accurate estimation, not only for the ellipse but for any (r),any astroid.

Use the following designations: Find (x)

Angle step =0,045
x=(Angle-45)/0.045
angle o=45+angle step*x
ATAN=angle o*Pi/180
angle o=ATAN*180/Pi

TAN=b/a=	5
ATAN=	1,373400767
Angle=	78,69006753 o
x=	748,66817

Use value (x) in Formulas and find

b2Mod=	0,288740099
m2Mod=	0,000997983
GMod=	1,948787387
pMod=	2,99518385
sMod=	1,567203335

Use sMod in L=(1+TAN^sMod)^(1/sMod)
Find L1estimated= 5,252513329792510
L1Exact=5,252511134922270 (with tools)
Control error %

Error %=(Lest-LExa)/Lest=0,000000417..

Nothing better than this result!
This is a world record.

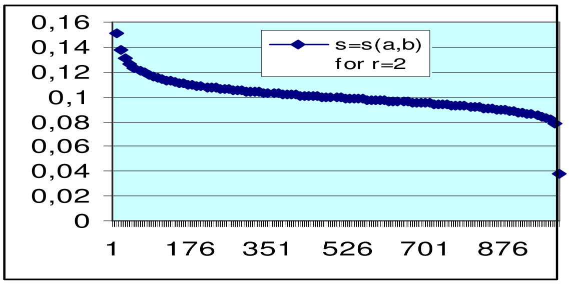
Conclusion

This *cracking* method is not only for the ellipse.
It is valid for all r (0<r<infinity)
It is valid for all TAN (1<TAN<infinity)

NB: Similar reasoning may be considered for AREA evaluation, with adequate math.

K^s=a^s+b^s will be considered
Say K=Area (as an object)
K/a will be the real area
K/a^3 will be the **constant** unit area

Example:(evaluation with 50 000 000 segments) **for**
r=2 (area of an ellipse is to be considered)
a=37
b=0,581242447..
That is b/a=TAN=0,0157..
s=0,15125741… extracted from (K^s)
K=624,9577492… =(a^s+b^s)^(1/s)
K/a=16,8907497627 =(1+TAN^s)^(1/s)
K/a^3=0,01233802… =**constant** unit area



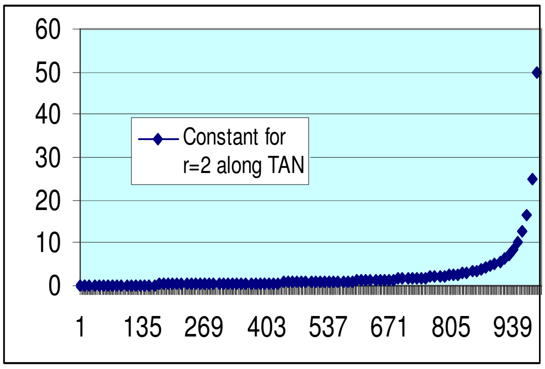
sGraph for r=2 (0<TAN<infinity).Not important, just typical.

The important is that
K/a^3=0,01233802… =**is a constant**
for (r=2) and (TAN=0,0157..) Then, the real area is expressed by
K/a=(K/a^3)*a^2
Example:

r=2
a=123
TAN=0,0157..=b/a
That is b=1,9311..
Real area=0,01233802..***123^2**
Real area=186,6619112…

r=2
a=238
TAN=0,0157..=b/a
Real area=0,01233802..***238^2**
Real area=698,8749097..

These mean that the constant area value is a function of (TAN) and (r). Here is its graph. This graph is important.



Application: (evaluation with 50 000 seg.)

r=2
a=2748
b=will vary in the following examples

b=43,1436..
TAN=0,0157..Var.Constant=0,01233802
Area=93171,79266..

b=259,686…
TAN=0,0945..Var.Constant=0,074242928
Area=560645,7662…

b=5393,2248…
TAN=1,9626..Var.Constant=1,541450261
Area=11640267,81….

Final remark : (for r=2)
Var.Constant/TAN=Coeff.**constant**
Coeff.Constant=0,785398163397448=Pi/4

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