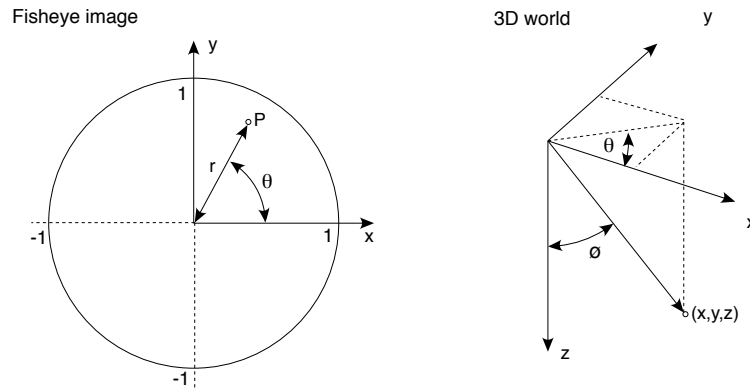**Fisheye mathematics**

Fisheye image   3D world

Any point P in a linear (mathematical) fisheye defines an angle of longitude and latitude and therefore a 3D vector into the world. Any real fisheye can be turned into a linear fisheye using a 1D correction polynomial.

θ = longitude = atan2(P.y,P.x)
ø = latitude = r $ø_{max}$ / 2   where $ø_{max}$ is the field of view of the fisheye lens

The vector into the world is given by

x = sin(ø) cos(θ)
y = sin(ø) sin(θ)
z = cos(ø)

Linearising a fisheye

The key to a fisheye is the relationship between latitude ø of the 3D vector and radius on the 2D fisheye image, namely a linear one where

$$ø(r) = r \; ø_{max}/2$$

Where r is the radius of the point on the fisheye in normalised coordinates (-1 to 1 across the fisheye circle) and $ø_{max}$ is the field of view of the fisheye across the fisheye circle. Note that this one dimensional function for what is a 2 dimensional curve works because fisheye lenses, like other lenses, are formed from a spinning process and are thus radially symmetric.
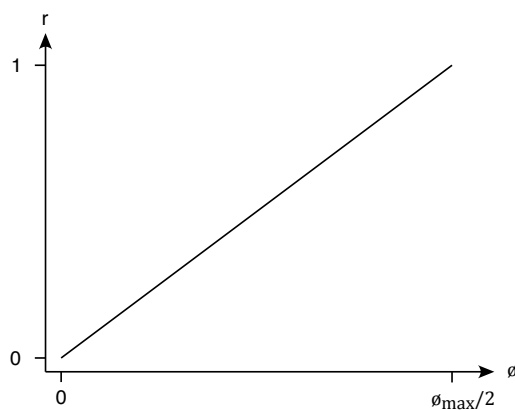
Where a real world fisheye deviates from the idealised relationship above is that real fisheye lenses are rarely (if ever) perfectly linear. Note a linear fisheye lens is often called a "true f-theta lens". So at some stage of any fisheye mapping process, assuming the lens is not a true f-theta lens (or a close enough approximation), is a function that maps real points on the fisheye to their position on an ideal fisheye.

There are two possible functions depending on whether one needs to convert normalised radii to actual latitude, or whether one needs to convert latitude to actual radius. In both cases a 4th order polynomial function is usually adequate. The form will be

$$ax + bx^2 + cx^3 + dx^4$$

Noting that since the origin of the curve always passes through the origin there is no constant term, put another way, r=0 is always latitude=0.

Illustrating this with an actual fisheye lens that is particularly nonlinear, the iZugar MKX22 220 degree fisheye. The method of measuring the points illustrated on these graphs will be the subject of the next section. In this case the horizontal axis is latitude ø and the vertical axis is normalised radius (0 to 1). Fitting a 4th order polynomial gives the following

$$r(ø) = 0.6622 \, ø - 0.0163 \, ø^2 + 0.0029 \, ø^3 - 0.0169 \, ø^4$$

In other words, given a latitude ø of a 3D vector the above gives the normalised radius on the fisheye. This is normally what is required for a image mapping of a fisheye into another image projection type, one computes the 3D vector for each pixel in the output image and the above gives the radius of the pixel in the real fisheye image.
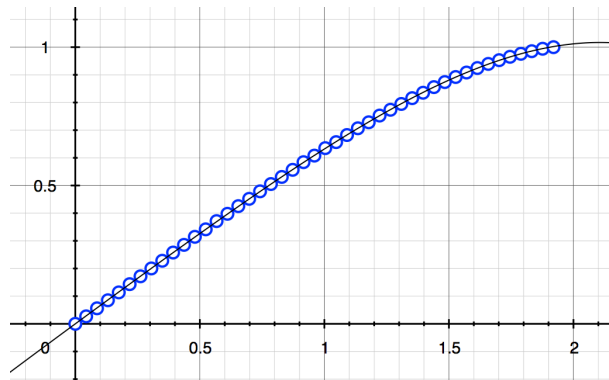
Sometime the other mapping is required, specifically, given a radius in the real fisheye image what is the latitude of the corresponding 3D vector. While in some cases one can invert the polynomial or solve the inverse numerically, much easier to just fit another polynomial to the swapped data. The following has normalised radius on the horizontal axis and latitude in radians on the vertical axis.
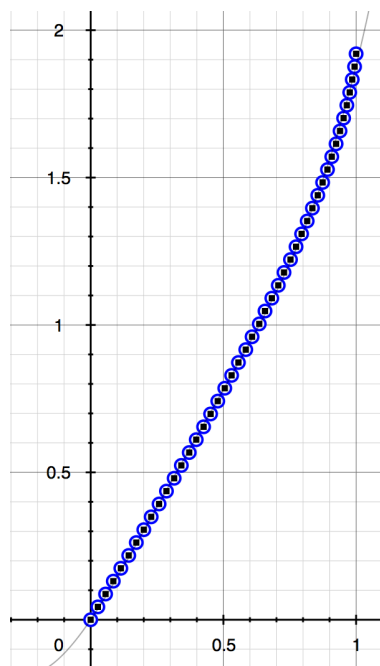
The fitted polynomial is as follows

$$\text{ø}(r) = 1.3202\ r + 1.4539\ r^2 - 2.9949\ r^3 + 2.1007\ r^4$$
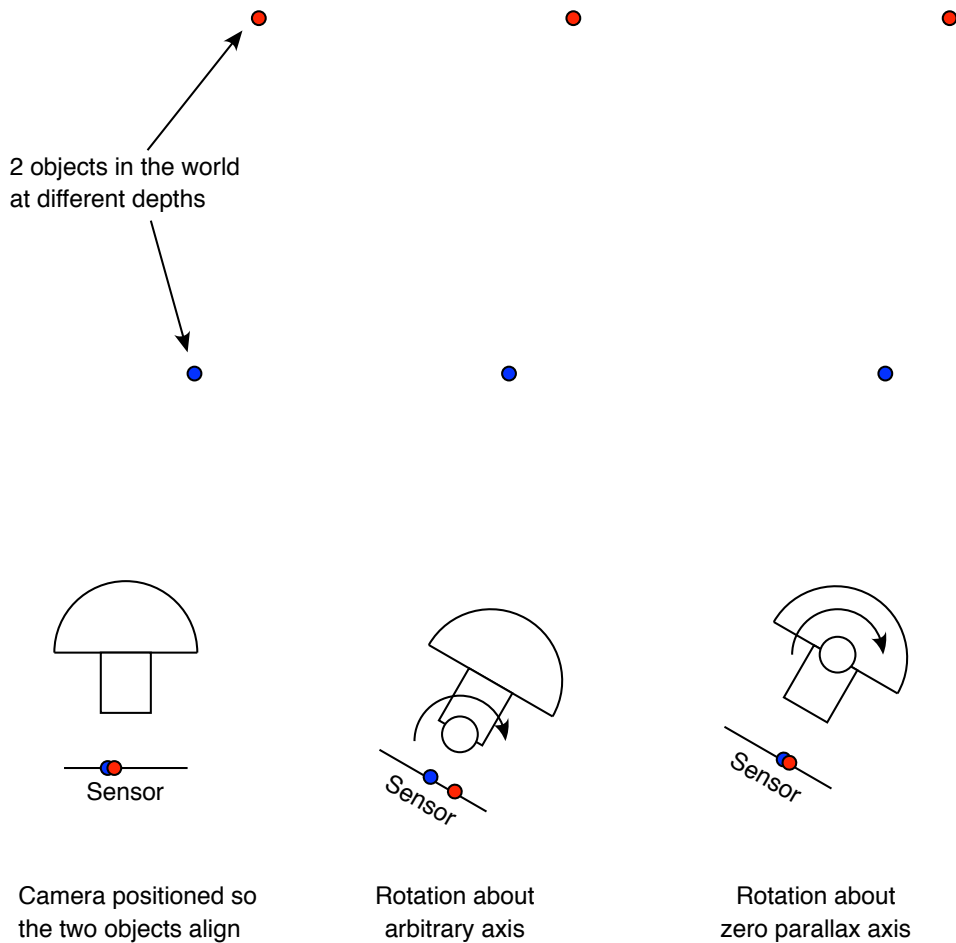
In other words, given a normalised radius r on the fisheye image this gives the true latitude ø. This is the function one normally needs if mapping individual points from the fisheye into 3D space.

Calculating linearising curves

In the previous section examples of the r, ø curves and how polynomials can be fit to those to either derive the correct r on the fisheye image given a latitude, or calculate the correct latitude given r on the fisheye image. There a number of ways of calculating the points for these curves, I will present two and conclude on what might be a practical and more precise alternative in your situation.

First I need to introduce the concept of the zero parallax point of a fisheye lens/camera system. If you are familiar with panorama photography then you may know it as the "nodal point". Rotating the lens about the zero parallax position ensures there will not be any stitching errors due to parallax. When doing measurements from a fisheye it is this zero parallax point that should be the origin.

The zero parallax point is typically located somewhere along the barrel of the fisheye lens. Sometimes it is marked on the metal barrel of the lens, but probably only for "professional" lenses. One can measure the zero parallax position by aligning up two objects in a scene that are at different distances, then rotating the lens/camera and inspecting the two objects on the image. At zero parallax the two objects will stay aligned, rotating about other positions will see the position of the near and far object separate. I've attempted to illustrate this below.

2 objects in the world
at different depths

Camera positioned so
the two objects align

Rotation about
arbitrary axis

Rotation about
zero parallax axis

Once the zero parallax position is known one can directly measure the points for the linearising curves. One way is with a rotating camera head, I use one with 5 degree increments.
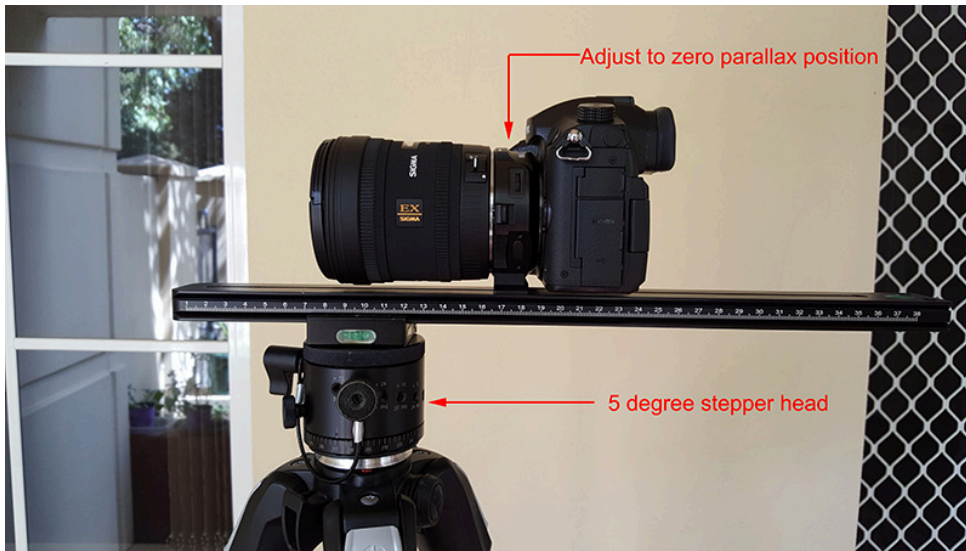
You choose a marker in the scene (eg: line on a wall) and rotate the camera/lens in equal angle steps and take a photograph. Later one measures the position of the marker on each photograph giving "r". Typically one first aligns the camera so the marker line is exactly center of the frame. One also typically rotates from $-\varnothing_{max}/2$ to $\varnothing_{max}/2$, for example

$y=0.7159x-0.0048x^2-0.032x^3+0.0021x^4$
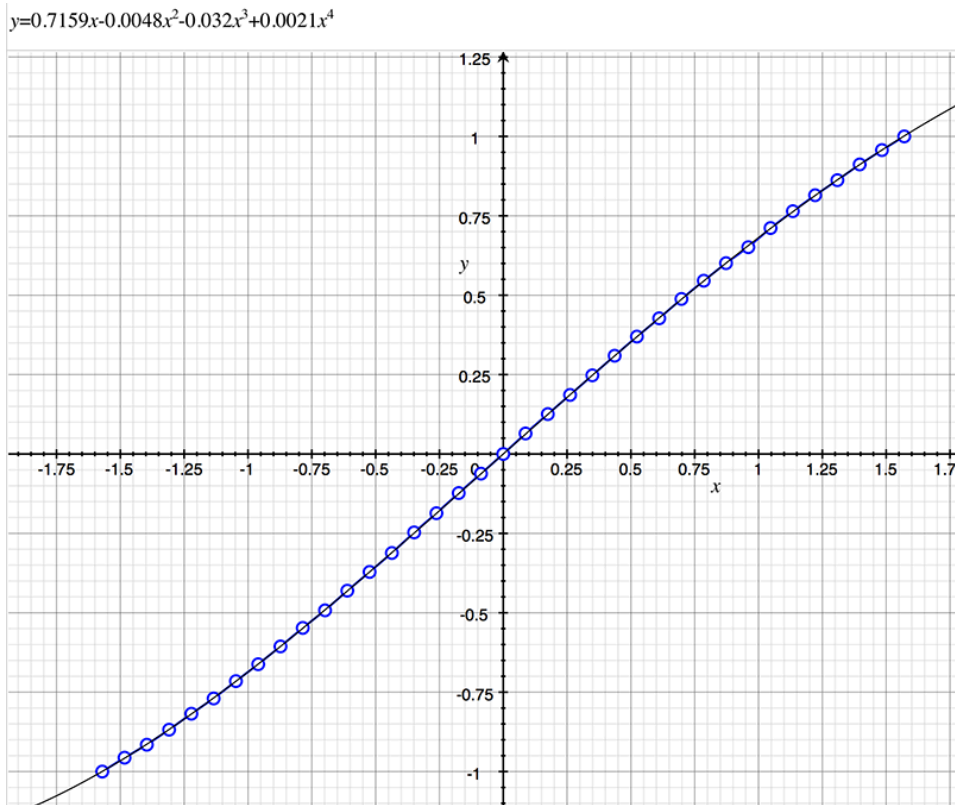
The other approach is to place a structure around the camera with markings of equal angle, or at least such that the angle can be measured. One reads off the angle markings on the camera image knowing their latitude in the real world.

Angle	Distance
0	
5	
10	
15	
FOV/2	
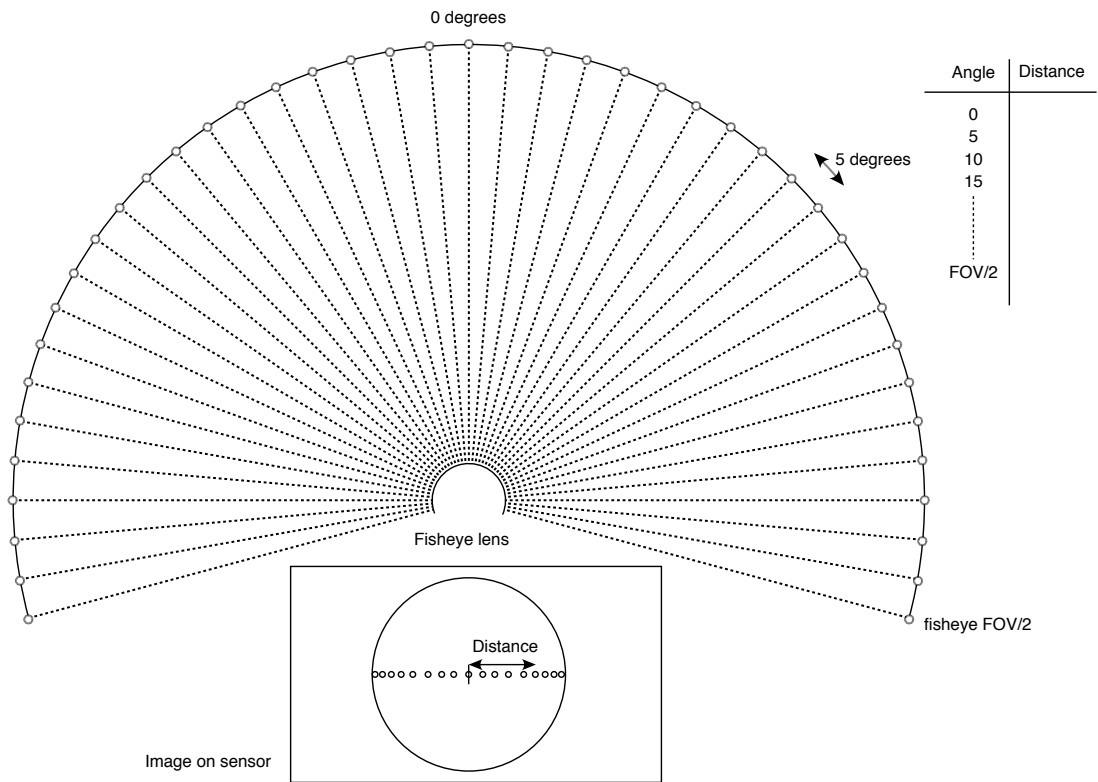
For example
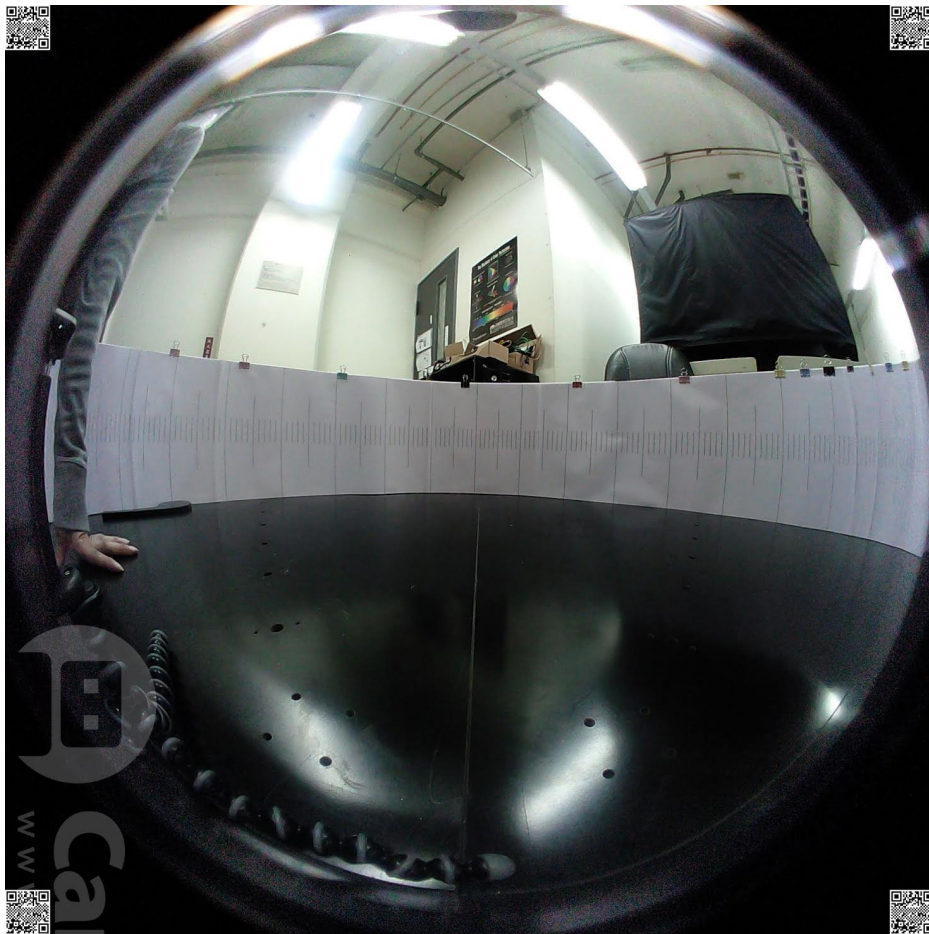
Another approach is to place markers at intervals along the floor and up opposite walls. The only real requirements is that the line of markers passes through an axis of the fisheye which in this example is pointing straight downwards from the ceiling.

Room side view

ø

x

z

Room top view
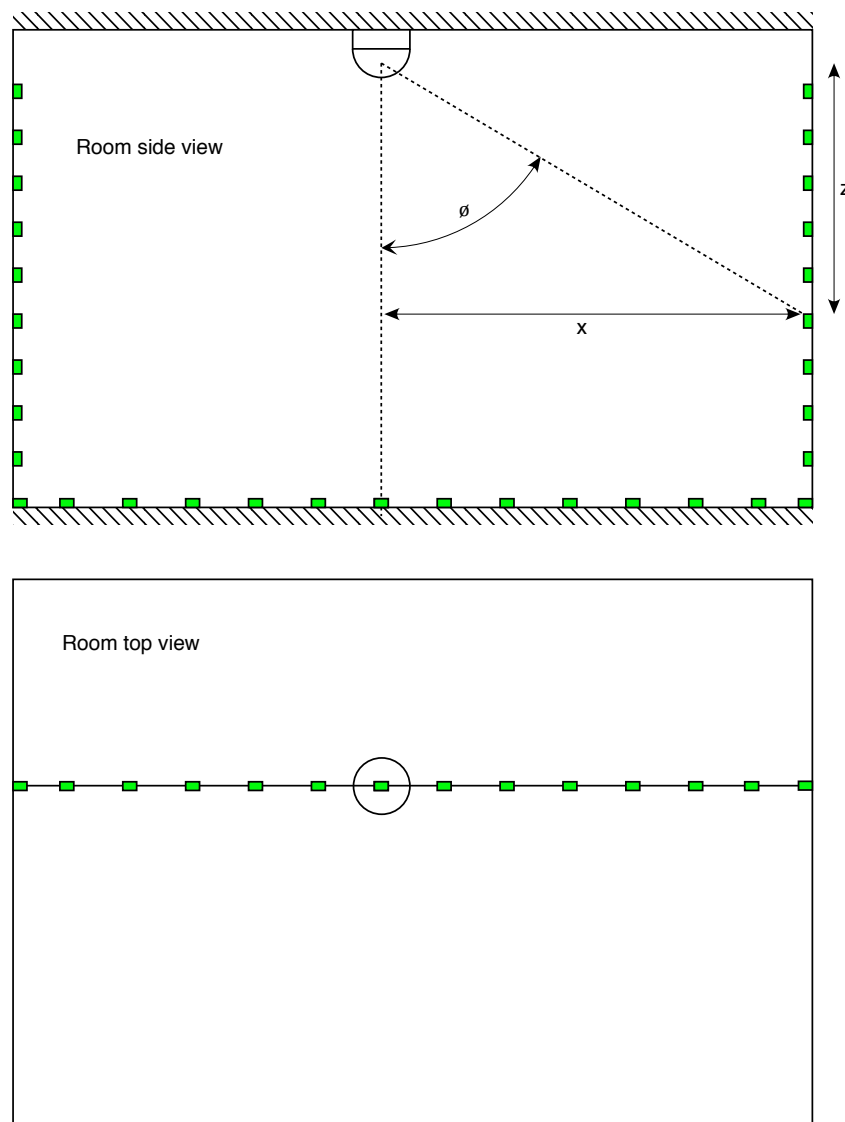
Measuring x and z allows you to calculate the latitude ø, one then just records the r value from the markers recorded in the camera image.

Where does all the zero parallax discussion come in? The origin for the measurements should be the zero parallax position on the lens barrel, but since one is often dealing with quite small lenses, choosing the front face of the lens or the front plate of the camera is probably only going to be out by a cm (or less).

As mentioned before, in theory you only need to do one half of the above, but doing both halves is a good test for symmetry (is the lens pointing straight down, are you measuring the center of the fisheye circle correctly, and so on).